Hedging and Calibration for Log-normal Rough Volatility Models

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Celebrating Jim Gatheral's 60th Birthday, 2017, New York

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- I was embarrassed, had to make an excuse for the model (this was just for a toy example, etc, etc).

Now this is a good memory for me.

The volatility skew power law

A figure from “Volatility is rough” by Gatheral et al. (2014).

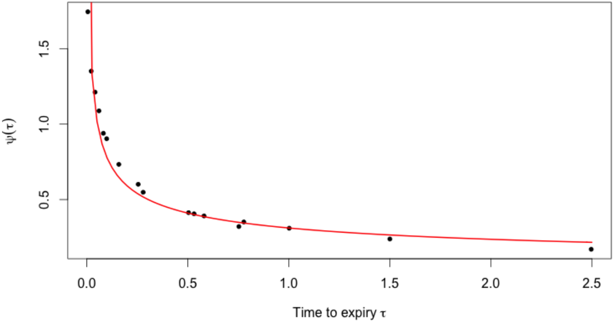


Figure 1.2: The black dots are non-parametric estimates of the S&P ATM volatility skews as of June 20, 2013; the red curve is the power-law fit $\psi(\tau) = A\,\tau^{-0.4}$.

Volatility is rough

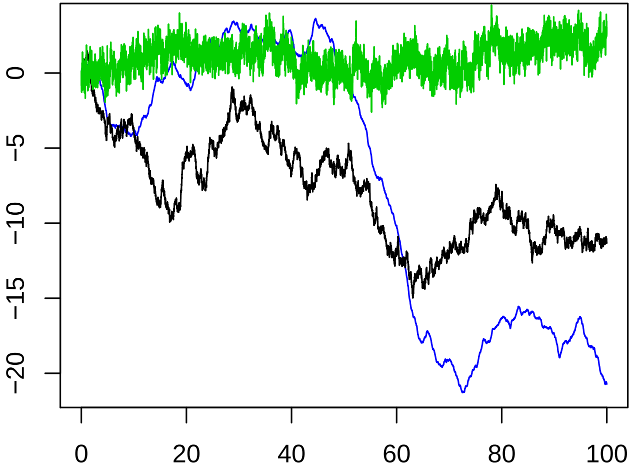
Gatheral, Jaisson and Rosenbaum (2014) showed that

- log realized variance increments exhibit a scaling property,
- a simple model

$$\mathrm{d}\langle \log S\rangle_t = V_t \mathrm{d}t, \quad \mathrm{d}\log V_t = \eta \mathrm{d}W_t^H$$

  is consistent to the scaling property with $H \approx .1$ as well as a stylized fact that the volatility is log normal,
- in particular, both the historical and implied volatilities suggest the same fractional volatility model $H \approx .1$,
- the model provides a good prediction performance,
- and the volatility paths from the model exhibit fake long memory properties.

fBm path: $H = 0.1, 0.5, 0.9$



Long memory and short memory

- The long memory property of asset return volatility originally meant a slow decay of the autocorrelation of squared returns.
- A mathematical definition is rigid; a stochastic process is of long memory iff its autocorrelation is not summable.
- In the case of fractional Gaussian noise $X_j = W^H_{j\Delta} - W^H_{(j-1)\Delta}$,

$$E[X_{j+k}X_j] = \frac{\Delta^{2H}}{2}(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}) \\ \sim \Delta^{2H} H(2H-1)k^{2H-2},$$

  so it is of long memory iff $H > 1/2$.
- In contrast, the case $H < 1/2$ is referred as being of short memory. It has by no means shorter memory than the case $H = 1/2$ that has no memory. The decay is actually slow.
- Set free from the long memory spell, goodbye bad memories.

Pricing under rough volatility

Bayer, Friz and Gatheral (2016) elegantly solved a pricing problem with "information from the big-bang":

- A fractional Brownian motion W^H is not Markov.
- The time t price of a payoff H is $E[H|\mathcal{F}_t]$ by no-arbitrage.
- The natural filtration of W^H is $\sigma(W_t^H - W_s^H; s \in (-\infty, t])$.

Rewrite the model under a martingale measure; for $\theta > t$

$$
\begin{aligned}
S_\theta &= S_t \exp\left\{ \int_t^\theta \sqrt{V_u} \mathrm{d}B_u - \frac{1}{2} \int_t^\theta V_u \mathrm{d}u \right\}, \\
V_\theta &= V_t \exp(\eta(W_\theta^H - W_t^H)) \\
&= V_t(\theta) \exp\left\{ \tilde{\eta} \int_t^\theta (\theta - u)^{H-1/2} \mathrm{d}W_u - \frac{\tilde{\eta}^2}{4H} (\theta - t)^{2H} \right\}
\end{aligned}
$$

and notice $E[\int_t^\theta \mathrm{d}\langle \log S \rangle_u | \mathcal{F}_t] = \int_t^\theta E[V_u|\mathcal{F}_t]\mathrm{d}u = \int_t^\theta V_t(u)\mathrm{d}u.$

The rough Bergomi model is Markov

The curve $\tau \mapsto V_t(t+\tau)$, where $V_t(\theta) =$

$$V_t \exp\left\{\tilde{\eta}\int_{-\infty}^{t}(\theta-u)^{H-1/2}-(t-u)^{H-1/2})\mathrm{d}W_u + \frac{\tilde{\eta}^2}{4H}(\theta-t)^{2H}\right\}$$

is called the forward variance curve. When $t > s$,

$$\begin{aligned} V_t(\theta) = V_s(\theta)\exp\Bigg\{&\tilde{\eta}\int_s^t(\theta-u)^{H-1/2}\mathrm{d}W_u \\ &- \frac{\tilde{\eta}^2}{4H}((\theta-s)^{2H}-(\theta-t)^{2H})\Bigg\}. \end{aligned}$$

Therefore the ∞ dimensional process

$$\{(S_t, V_t(t+\cdot))\}_{t\geq 0}$$

is Markov with $(0,\infty)\times C([0,\infty))$ as its state space.

An extension: log-normal rough volatility models

The rough Bergomi model of BFG can be written as

$$\begin{aligned} S_\theta &= S_t \exp\left\{ \int_t^\theta \sqrt{V_u} \mathrm{d}B_u - \frac{1}{2} \int_t^\theta V_u \mathrm{d}u \right\}, \\ V_\theta &= V_t(\theta) \exp\left\{ \int_t^\theta k(\theta,u) \mathrm{d}W_u - \frac{1}{2} \int_t^\theta k(\theta,u)^2 \mathrm{d}u \right\}, \\ V_t(\theta) &= V_s(\theta) \exp\left\{ \int_s^t k(\theta,u) \mathrm{d}W_u - \frac{1}{2} \int_s^t k(\theta,u)^2 \mathrm{d}u \right\} \end{aligned}$$

for $\theta > t > s$ with $k(\theta, u) = \tilde{\eta}(\theta - u)^{H-1/2}$ and $\mathrm{d}\langle B, W \rangle_t = \rho \mathrm{d}t$.

Notice the forward variance curve follows time-inhomogeneous Black-Scholes; for each θ,

$$\mathrm{d}V_t(\theta) = V_t(\theta) k(\theta, t) \mathrm{d}W_t, \quad t < \theta.$$

Log-contract price dynamics

$$
\begin{aligned}
E[-2\log S_\theta|\mathcal{F}_t] &= -2\log S_t + E[\int_t^\theta \mathrm{d}\langle \log S\rangle_u|\mathcal{F}_t] \\
&= -2\log S_t + \int_t^\theta V_t(u)\mathrm{d}u \\
&= -2\log S_0 - 2\int_0^t \frac{\mathrm{d}S_u}{S_u} + \int_0^t V_u \mathrm{d}u + \int_t^\theta V_t(u)\mathrm{d}u.
\end{aligned}
$$

Therefore, $P_t^\theta = E[-2\log S_\theta|\mathcal{F}_t]$ follows

$$
\begin{aligned}
\mathrm{d}P_t^\theta &= -2\frac{\mathrm{d}S_t}{S_t} + \int_t^\theta \mathrm{d}V_t(u)\mathrm{d}u \\
&= -2\frac{\mathrm{d}S_t}{S_t} + \left\{\int_t^\theta V_t(u)k(u,t)\mathrm{d}u\right\}\mathrm{d}W_t \\
&= -2\frac{\mathrm{d}S_t}{S_t} + \left\{\int_t^\theta \frac{\partial P_t^u}{\partial u}k(u,t)\mathrm{d}u\right\}\mathrm{d}W_t.
\end{aligned}
$$

Hedging under rough volatility

Theorem. Let P^θ be a log-contract price process with maturity θ. Then, any square-integrable payoff with maturity $\tau \le \theta$ can be perfectly replicated by a dynamic portfolio of (S, P^θ).

Proof. Write $B = \rho W + \sqrt{1-\rho^2} W^\perp$. Then, the martingale representation theorem tells that for any X there exists $(H, H^\perp)$ such that

$$X = E[X|\mathcal{F}_0] + \int_0^\tau H_t \mathrm{d}W_t + \int_0^\tau H_t^\perp \mathrm{d}W_t^\perp.$$

(Use the Clark-Ocone to compute it). We have

$$\mathrm{d}W_t^\perp = \frac{1}{\sqrt{1-\rho^2}} \left\{ \frac{\mathrm{d}S_t}{\sqrt{V_t} S_t} - \rho \mathrm{d}W_t \right\}$$

$$\mathrm{d}W_t = \left\{ \int_t^\theta \frac{\partial P_t^u}{\partial u} k(u,t) \mathrm{d}u \right\}^{-1} \left\{ \mathrm{d}P_t^\theta + 2\frac{\mathrm{d}S_t}{S_t} \right\}.$$

An example

Consider to hedge a log-contract with maturity τ by one with $\theta > \tau$. Using again

$$\mathrm{d}P_t^\theta = -2\frac{\mathrm{d}S_t}{S_t} + \left\{ \int_t^\theta \frac{\partial P_t^u}{\partial u} k(u,t)\mathrm{d}u \right\} \mathrm{d}W_t,$$

we have

$$\begin{aligned} \mathrm{d}P_t^\tau &= -2\frac{\mathrm{d}S_t}{S_t} + \left\{ \int_t^\tau \frac{\partial P_t^u}{\partial u} k(u,t)\mathrm{d}u \right\} \mathrm{d}W_t \\ &= -2\frac{\mathrm{d}S_t}{S_t} + \frac{\int_t^\tau \frac{\partial P_t^u}{\partial u} k(u,t)\mathrm{d}u}{\int_t^\theta \frac{\partial P_t^u}{\partial u} k(u,t)\mathrm{d}u} \left\{ \mathrm{d}P_t^\theta + 2\frac{\mathrm{d}S_t}{S_t} \right\}. \end{aligned}$$

Consistent to real market data ?

A related ongoing work: Horvath, Jacquier and Tankov.

How to calibrate ?

Monte Carlo $\to$ The next talk !

Asymptotic analyses under flat (or specific) forward variances:

- Alòs et al (2007)
- Fukasawa (2011)
- Bayer, Friz and Gatheral (2016)
- Forde and Zhang (2017)
- Jacquier, Pakkanen, Stone
- Bayer, Friz, Gulisashvili, Horvath, Stemper
- Akahori, Song, Wang
- Funahashi and Kijima (2017) and more.

Asymptotic analyses under a general forward variance curve:

- Fukasawa (2017)
- Garnier and Solna
- El Euch, Fukasawa, Gatheral and Rosenbaum (in preparation)

The ATM implied volatility skew and curvature

El Euch, Fukasawa, Gatheral and Rosenbaum: as $\theta \to 0$,

$$\sigma_t(0,\theta) = \left\{1 + \left(\frac{3\kappa_3^2}{2} - \kappa_4\right)\theta^{2H}\right\}\sqrt{\frac{1}{\theta}\int_0^\theta V_t(t+\tau)\mathrm{d}\tau} + o(\theta^{2H}),$$

$$\frac{\partial}{\partial k}\sigma_t(k,\theta)\bigg|_{k=0} = \kappa_3\theta^{H-1/2} + o(\theta^{2H-1/2}),$$

$$\frac{\partial^2}{\partial k^2}\sigma_t(k,\theta)\bigg|_{k=0} = 2\frac{\kappa_4 - 3\kappa_3^2}{\sqrt{V_t}}\theta^{2H-1} + \kappa_3\theta^{H-1/2} + o(\theta^{2H-1}),$$

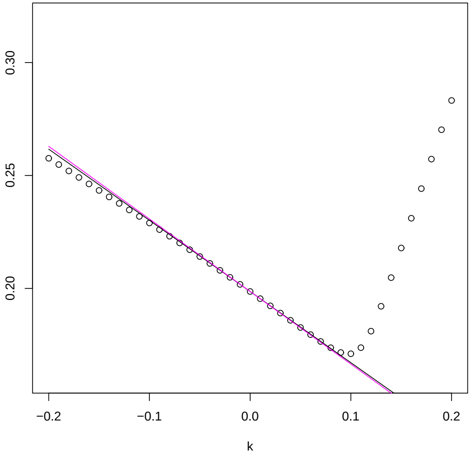
under the rough Bergomi model with $|\rho| < 1$ and forward variance curve of H-Hölder, where

$$\kappa_3 = \frac{\rho\tilde{\eta}}{2(H+1/2)(H+3/2)},$$

$$\kappa_4 = \frac{(1+2\rho^2)\tilde{\eta}^2}{4(H+1)(2H+1)^2} + \frac{\rho^2\tilde{\eta}^2\beta(H+3/2,H+3/2)}{(2H+1)^2}.$$

$H = .05,\ \rho = -.9,\ \frac{\tilde{\eta}}{\sqrt{2H}} = .5,\ V(0) = .04,\ \theta = 1$, flat

$$\frac{\tilde{\eta}}{\sqrt{2H}}\theta^H < 1.$$



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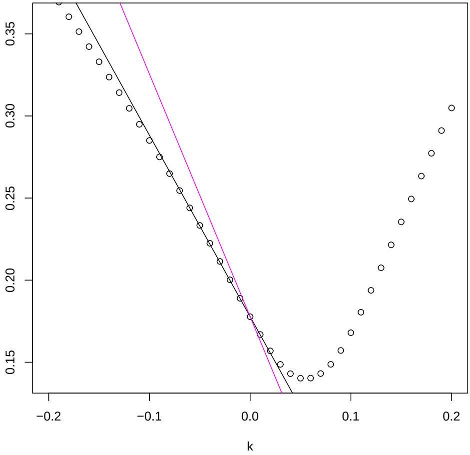


An intermediate formula

Let $t = 0$ for simplicity.

Theorem.

$$\frac{\partial}{\partial k}\sigma_0(k,\theta)\Big|_{k=0} \sim -\frac{\rho}{\sqrt{\theta}} E\left[\frac{X_\theta}{\sqrt{\langle X\rangle_\theta}}\right]$$

as $\theta \to 0$, where

$$X_\theta = \int_0^\theta \sqrt{V_s}\,\mathrm{d}W_s,$$

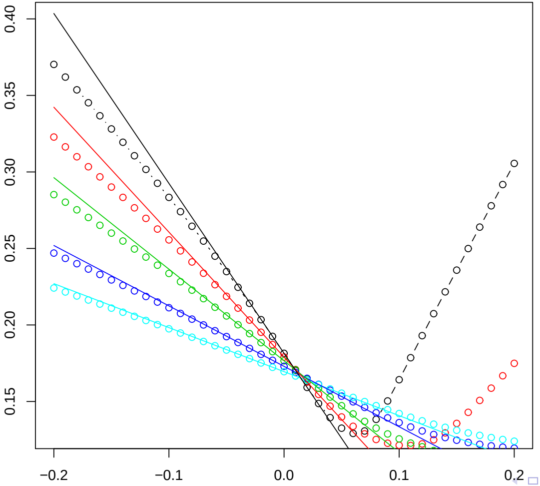
$$V_s = V_0(s)\exp\left\{\int_0^s k(s,u)\,\mathrm{d}W_u - \frac{1}{2}\int_0^s k(s,u)^2\,\mathrm{d}u\right\}.$$

Note: we still need Monte-Carlo, but it is free from ρ.

This approximation is surprisingly accurate !

$$H = .07,\ \rho = -.9,\ \frac{\tilde{\eta}}{\sqrt{2H}} = 1.9,\ V(0) = .04,\ \text{flat}$$

$\theta = 0.05, 0.1, 0.2, 0.5, 1.0$



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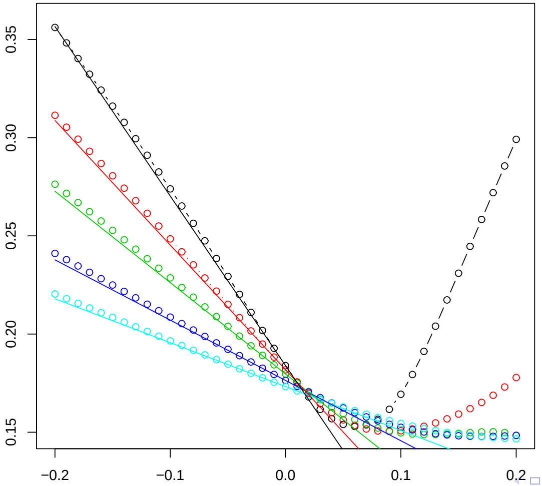

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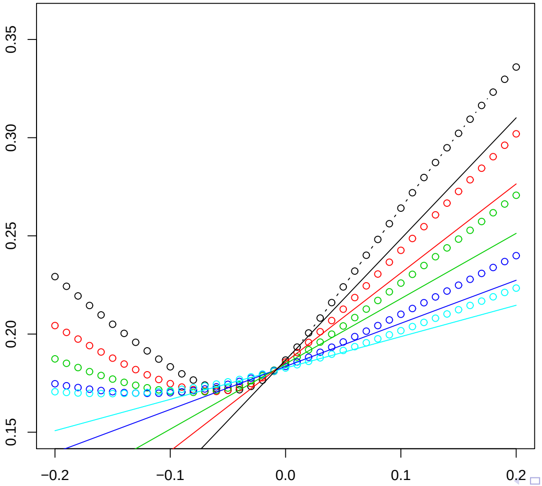

$$H = .05,\ \rho = -.9,\ \frac{\tilde{\eta}}{\sqrt{2H}} = 2.3,\ V(0) = .04,\ \text{flat}$$

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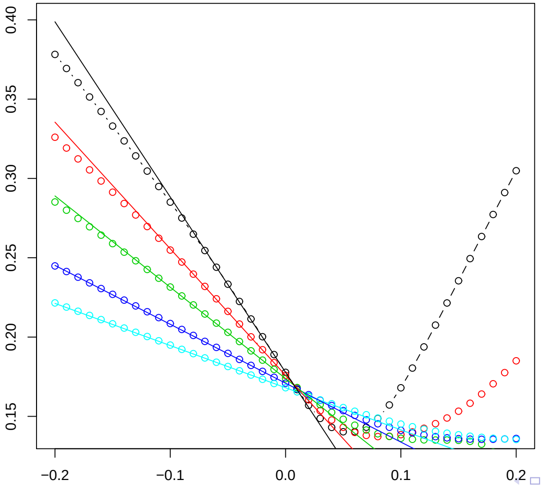

$$H = .07,\ \rho = -.9,\ \frac{\tilde{\eta}}{\sqrt{2H}} = 1.9,\ V(0) = .04,\ \text{sin}$$

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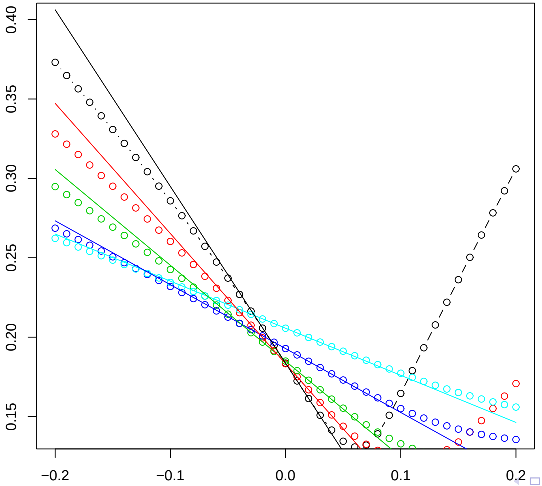

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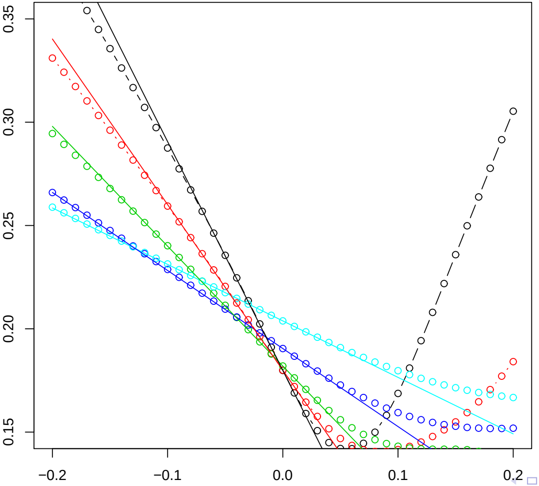

$$H = .05,\ \rho = -.9,\ \frac{\tilde{\eta}}{\sqrt{2H}} = 5.0,\ V(0) = .04,\ \text{flat}$$

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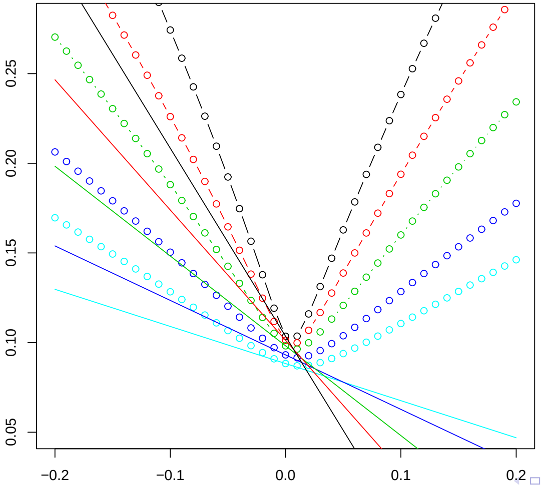
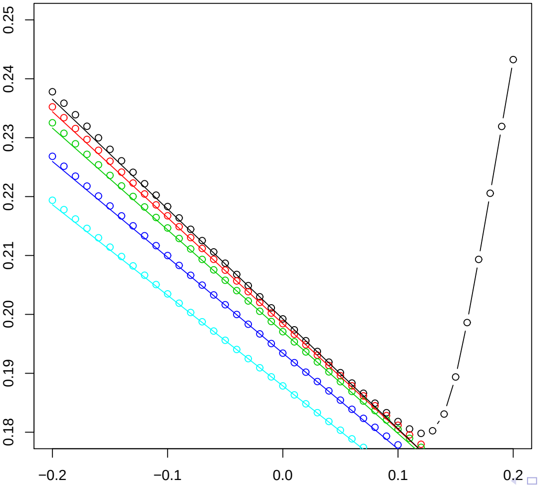


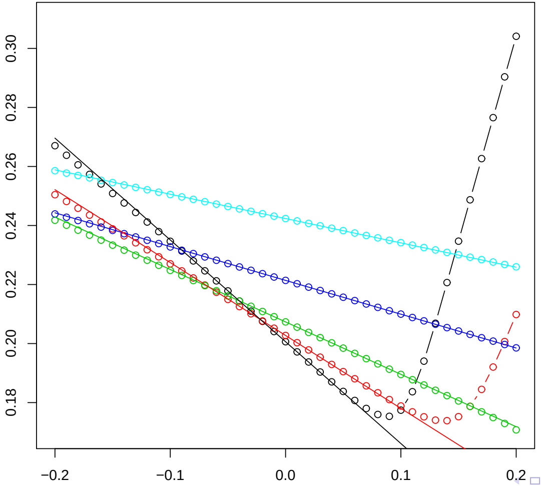
$H = .45,\ \rho = -.7,\ \frac{\tilde{\eta}}{\sqrt{2H}} = .9,\ V(0) = .04,$ flat

$\theta = 0.05, 0.1, 0.2, 0.5, 1.0$



$H = .01,\ \rho = -.9,\ \frac{\tilde{\eta}}{\sqrt{2H}} = 1.1,\ V(0) = .04,$ flat

$\theta = 0.05, 0.1, 0.2, 0.5, 1.0$



Conclusion

The (log-normal) rough volatility is very attractive

- mathematical structure
- impressive fit to the volatility surface

There are still mysteries...

- why is the slope formula so accurate ?
- why is volatility rough ?

More mathematical questions

- the critical moment ?
- limit distribution of discretization error ?
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Research will go on.

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Congratulations Jim and cheers to your model !!